

From the Hosfadtter to the Fibonacci butterfly

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We show that the electronic spectrum of a tight-binding hamiltonian defined in a quasiperiodic chain with an on-site potential given by a Fibonacci sequence, can be obtained as a superposition of Harper potentials. The electronic spectrum of the Harper equation is a fractal set, known as Hosfadtter butterfly. Here we show that is possible to construct a similar butterfly for the Fibonacci potential just by adding harmonics to the Harper potential. As a result, the equations in reciprocal space for the Fibonacci case have the form of a chain with a long range interaction between Fourier components. Then we explore the transformation between both spectra, and specially the origin of energy gaps due to the analytical calculation of the components in reciprocal space of the potentials. We also calculate some localization properties by finding the correlator of each potential.

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I. INTRODUCTION

Although the discovery of quasicrystals [1], which are alloys with an structure neither periodic, nor disordered, launched an extensive investigation on quasiperiodic Hamiltonians, in fact the study of such Hamiltonians goes back to the old Frenkel-Kontorova model [2] and to the pioneer research made in the seventies [3, 4]. Here the word quasiperiodic means that in the system there are incommensurate periods; and as a result, the dimension of the Fourier space is always bigger than the dimensionality of the system. One of the most famous quasiperiodic Hamiltonians was obtained by Harper in connection with a problem proposed by Peierls [3]. The idea was to find the spectrum and the wave-functions of an electron in a square lattice with a perpendicular magnetic field. Two periods are involved in the problem, the electron motion in the lattice and the cyclotron frequency [3]. The spectrum as a function of ratio between these periods turned out to be a complex set known as the Hostadter butterfly [3] (see figure 1). Since then, the Harper model has been very useful to investigate the transition from localized to extended eigenstates, as the spectrum pass from pure point to continuous [3][4][5]. Between both limits, there is a new type of spectrum which is known as singular continuous [4]. The corresponding eigenstates are called critical and show self-similar properties. For certain parameters of the Harper equation, the distribution of level spacings follows an inverse power law [6], which is a new type of spectral statistics [6], explained as a level clustering tendency [7]. It has been possible to find analytical expressions for the wave-functions using quantum groups [8]. More recently, the quantum phase diagrams [9] and the electronic correlation effects have been analyzed [10].

Another quasiperiodic system that has been extensively studied is the Fibonacci chain (FC). This chain is the simplest model of a quasicrystal [11]. The importance of this Hamiltonian arises because the nature of the physical properties of quasicrystals is still not well under-

stood [12][13][14]. Even in the theoretical side there is a lack of understanding in how electrons propagate, specially in two and three dimensions [15]. As is well known, a periodic potential satisfies the Bloch's theorem, which tells that the eigenstates of the Schrödinger equation are plane waves of delocalized nature, and the energy spectrum is continuous [16]. For disordered systems, like in the one dimensional (1D) Anderson model, all the states are localized corresponding to isolated eigenvalues [17]. In more dimensions, there is a mobility edge which separates extended from localized states [17]. For most of the quasiperiodic systems in 1D, the spectrum is neither continuous nor singular, instead a new type of spectrum, called singular continuous is obtained [18]. This kind of spectrum is similar to a Cantor set, and presents a multifractal nature. The corresponding eigenfunctions are called critical, and also show self-similarity and multifractality. In two and three dimensions, the nature of the spectrum is not known, although there seems to be a kind of mobility edge [19][20]. However, even in 1D, where large amount of work has been done, there are many unsolved questions, like the nature of conductivity [21] or diffusivity [5], the spectral statistics and the shape of many of the eigenfunctions [22][23]. Even in the FC, there are no analytical expression for the wave functions, except for few energies [11].

The Harper and Fibonacci potentials share many characteristics; for example, both can present a multifractal spectrum with self-similar wave functions, although the Harper equation can also present pure point and continuous spectrum. An interesting question is why Fibonacci does not show pure point or continuous spectrum. Furthermore, is it possible to build the analogous of the Hostadter butterfly for the FC? An understanding of these questions will serve to give a better picture of the electronic properties of quasiperiodic systems. For example, it can suggest a way to construct analytical solutions for the FC in terms of those solutions from Harper.

In this article, we show that in fact, the Fibonacci potential can be made from a superposition of Harper

potentials. Then, we can follow the transition of the Hofstadter butterfly to the equivalent in the Fibonacci case. This allows to explore the equations in reciprocal space of the FC. The layout of this work is the following, in section II we show how to obtain the Fibonacci potential in terms of Harper. Section III is devoted to a discussion of the corresponding spectra using the properties in reciprocal space, while section IV is devoted to study the localization in terms of the correlators of the potential. Finally, in section V the conclusions are given.

II. THE FIBONACCI AND HARPER MODELS

As a general model we will use a tight-binding Hamiltonian of the type,

$$(E - V(n))\psi_n = t_n\psi_{n+1} + t_{n-1}\psi_{n-1}, \quad (1)$$

where ψ_n is the wave-function at site n , t_n is the resonance integral between sites n and $n+1$. For the present purposes, t_n is set to 1 for all sites. $V(n)$ is the atomic on-site potential and E are the allowed energies. The generalized Harper equation is obtained when [3],

$$V(n) \equiv V_H(n) \equiv 2\lambda \cos(2\pi\phi n + v), \quad (2)$$

where $\lambda \geq 0$ is the strength of the potential, ϕ is a parameter that contains the ratio between the electron cyclotron frequency and the elementary quantum flux, and v is a phase shift. For a rational ϕ , Eq. (1) can be solved by Bloch's theorem since the problem is periodic, although the value of such theorem is very limited since the coefficients in Fourier space of the solution form a dense set [3]. For ϕ irrational, the spectrum depends on the value of λ . For $\lambda < 1$, the spectrum is continuous with extended wave functions, for $\lambda > 1$ the spectrum is made from pure points and localized solutions. For $\lambda = 1$ the spectrum is singular continuous with self-similar wave functions.

The other potential that we will consider, is the simplest model of a quasicrystal, obtained when $V(n)$ has two possible values, V_A and V_B following the Fibonacci sequence (FS). The FS is build as follows: consider two letters, A and B , and the substitution rules, $A \rightarrow B$, and $B \rightarrow AB$. If one defines the first generation sequence as $\mathcal{F}_1 = A$ and the second one as $\mathcal{F}_2 = BA$, the subsequent chains are generated using the two previous rules, for instance, $\mathcal{F}_3 = ABA$. Starting with an A , we construct the following sequences, A , B , AB , BAB , $ABBAB$, $BABABBAB$, and so on. Each generation obtained by iteration of the rules is labeled with an index l . Is clear that the number of letters in each generation l is given by the Fibonacci numbers $F(l)$ of generation l , which satisfy: $F(l) = F(l-1) + F(l-2)$ with the initial conditions: $F(0) = 1, F(1) = 1$. A Fibonacci potential is obtained with two values V_A, V_B which follow a FS.

Our first task in order to compare the FC and the Harper potential, consists in finding an analytical expression for the Fibonacci potential. This can be done in the following way. By using the cut and projection method, it is very easy to prove that a one dimensional chain with atoms at positions y_n determined by a FS are given by [24],

$$y_n = [n\phi]$$

where the function $[x]$ denotes the greatest integer lower than x , ϕ is a parameter which turns out to be the inverse of the golden mean $\tau^{-1} = (\sqrt{5} - 1)/2$. The separation between atoms is $y_{n+1} - y_n$. In the Fibonacci sequence, the separation takes two values, l_A and l_B . Thus, we can make a Fibonacci potential with the sequence of spacings to get,

$$V(n) = V_B + V_A ([(n+1)\phi] - [n\phi]), \quad (3)$$

Using the identity $x = [x] + \{x\}$, where $\{x\}$ is the decimal part of x , we obtain that $V(n)$ can be written as,

$$V(n) = \langle V \rangle + \delta V (\{n\phi\} - \{(n+1)\phi\}), \quad (4)$$

where $\langle V \rangle = V_A\phi + V_B(1-\phi)$ is an average potential that shifts the zero of the energies, and δV is the strength of the quasiperiodicity, measured by the difference between site-energies $\delta V = V_A - V_B$. In what follows, without any loss of generality we set V_A and V_B in such a way that $\langle V \rangle = 0$. Now, if the integer variable n is replaced by a continuous one, say again x , the resulting potential $V(x)$ is just a square wave as shown in figure 1. The potential stays at $-\delta V\phi$ for an interval of length $1 - \phi$, and then it jumps to $\delta V(1 - \phi)$ for a length ϕ .

The decimal part function $\{x\}$ has period 1, and can be developed as a Fourier series,

$$\{x\phi\} = \frac{1}{2} - \frac{1}{\pi} \sum_{s=1}^{\infty} \frac{1}{s} \sin(2\pi\phi s x),$$

It follows that,

$$V(n) = \bar{V} + 2\delta V \sum_{s=1}^{\infty} \tilde{V}(s) \cos(\pi s \phi (2n+1)), \quad (5)$$

where $\tilde{V}(s)$ is the s harmonic of the Fourier series, $\tilde{V}(s) = \sin(\pi s \phi)/\pi s$. The first terms of this series are shown in figure 1, and in fact, we are approximating a square wave by a sum of cosines. Notice the very slow convergence of the series due to the $1/s$ factor of each harmonic. This potential can be further reduced if a proper phase χ is used in Eq. (4), in such a way that the terms $\{x\phi\}$ are replaced by $\{x\phi + \chi\}$. This phase is only a horizontal translation of the potential. For $\chi = -\phi/2$, the Fibonacci potential is simply written as,

$$V(n) = 2\delta V \sum_{s=1}^{\infty} \tilde{V}(s) \cos(2\pi s \phi n), \quad (6)$$

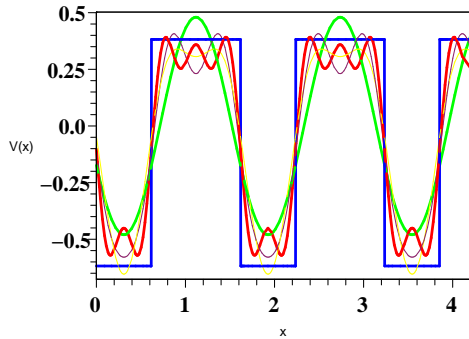


FIG. 1: The Fibonacci potential can be obtained by evaluating a square wave of period τ at integers values. The approximation with one, two, three and four harmonics, obtained from Eq. (6), are shown in the figure.

It is also worthwhile mentioning that for $s = l\phi^{-1}$ (where l is an integer), $\tilde{V}(s) = 0$. This condition can only hold when ϕ is a rational. When this is not the case, $\tilde{V}(s) \approx 0$ for integers s and l such that $\phi \approx l/s$, and thus l/s is a rational approximant of ϕ . Such rationals are obtained from the continuous fraction development of the number ϕ . For the FS, $s \approx l\tau$ from where it follows that l and s are successive Fibonacci numbers. Such condition can be interpreted as the existence of a rational approximant structure when ϕ is set to $\phi = l/s$. The most important Fourier components in Eq. (6), are those harmonics for which $s \approx (r + 1/2)\phi^{-1}$ for an integer r . Using the decomposition in integer and decimal parts, this happens whenever $\{(r + 1/2)\phi^{-1}\}$ is nearly zero or one.

Comparing eq.(2) with eq.(5) and eq.(6), we observe that in fact, the Fibonacci potential is just a superposition of Harper potentials with the right amplitude and phase. In some sense, this is a similar situation to an applied effective modulated magnetic field [25]. This leads to many questions. The first is how the transition from Harper is done. To answer this, we will cut the sum in Eq. (6) at a finite number of harmonics, denoted by S . In figure 2 a) we plot the energy spectrum of the Harper equation in the case $\lambda = 1$. This spectrum, as well as the others discussed in this article, were obtained by using the transfer matrix formalism [26]. Figure 1 a) is the well known Hofstadter butterfly [3]. Notice that here $v = \pi\phi$, and thus our figure is not exactly the same as in Hofstadter work, since therein, a sweep for all values of v between 0 and 2π is made, and thus the spectrum has more points.

If ϕ is taken as a parameter in eq.(6), then the spectrum of Fig. 2a) is also the same as for the Fibonacci potential but with only one harmonic, $s = 1$. The parameter used is $\delta V = \pi/\sin(\pi\tau^{-1}) \approx 3.3706$, chosen to correspond to $\lambda = 1$ in the Harper equation. In Figure

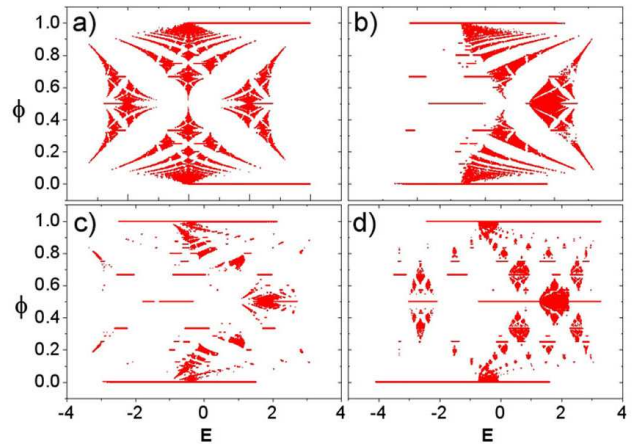


FIG. 2: The energy spectrum as a function of the parameter ϕ for $\delta V = \pi/\sin(\pi\tau^{-1})$, corresponding to $\lambda = 1$ in the pure Harper equation, using a) one harmonic (Hofstadter butterfly), b) two harmonics, c) three harmonics, and d) the full Fibonacci potential ("Fibonacci butterfly").

2b) and 2c) we show the effects of adding harmonics $s = 2$ and $s = 3$ in the development of the potential, and figure 2d) presents the result for the exact FC. There is an important change between the pure Harper case and the second harmonic case. Mainly the left part of the Hofstadter butterfly is washed out. Also, is clear that with only three harmonics, the structure is already very similar to the pure Fibonacci case. This spectacular wash out due to the second harmonic has its origins in the difference of lengths in the steps of the square wave Fibonacci potential. For the first harmonic, this change is not observed. But by looking at Fig. 1, one can observe that with two harmonics there is an asymmetry in the upper part of the series. This effect is more notorious when $\delta V > 1$, and can be translated in an almost split band limit when $\delta V \rightarrow \infty$, around self-energies V_A and V_B . In fact, this is one of the main differences between the Harper and Fibonacci potentials.

In Figure 3, a similar set of figures presents what happens when $\delta V = 1$. This corresponds to $\lambda = \sin(\pi\tau^{-1}) \approx 0.2967$ in the pure Harper case, and the spectrum is continuous. Figure 3b) and 3c) are the cases with two and three harmonics. Finally Fig. 3d) shows the case of the pure Fibonacci chain, which is a beautiful fractal. Since it is known that the FC presents a singular continuous spectrum [18], it is open the question for which harmonic there is the transition from one type of spectrum to the other.

In all of the previous cases, ϕ was studied between 0 and 1 since using the following identity,

$$\cos(2\pi x) = \cos(2\pi(\lfloor x \rfloor + \{x\})) = \cos(2\pi\{x\}),$$

we have that $\cos(2\pi\phi sn + k) = \cos(2\pi\{\phi\}sn + k)$. Thus, the problem has periodicity 1 in ϕ , since it depends on ϕ

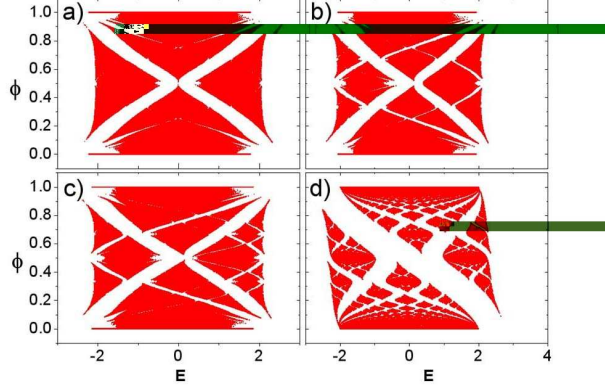


FIG. 3: The energy spectrum as a function of the parameter ϕ for $\delta V = 1$ using a) one harmonic (corresponding to the Harper model at $\lambda = 0.2967$), b) two harmonics, c) three harmonics, and d) full Fibonacci potential.

only through $\{\phi\}$. For rational ϕ of the form P/Q , with P and Q integers, this means $P < Q$.

III. DISCUSSION

In this section we will discuss the main features that arises from the previous figures. The spectral properties of the pure Hofstadter butterfly has been discussed by many others using diverse techniques [3][4][27], but in order to understand the transition between butterflies, here we will explain the main features using the structure in reciprocal space of the potential. Let us first study the tight-binding equation in the usual approach. We propose that the wave-function can be written as [4],

$$\psi_n = e^{ikn} \sum_{m=-\infty}^{\infty} d_m e^{im(2\pi\phi n + v)},$$

where d_m is the m Fourier component of the wave function. If we introduce this solution into Eq. (1), when S harmonics are present in the potential, the following set of equations are obtained,

$$(E^* - 2\delta V^* \cos(2\pi\phi m + k)) d_m = d_{m+1} + d_{m-1} + \quad (7)$$

$$\sum_{s=2}^S \tilde{V}^*(s) (d_{m-s} + d_{m+s}), \quad (8)$$

where the parameters are defined as $E^* = 2E/(V(1)\delta V)$, $\delta V^* = 2/(V(1)\delta V)$, and,

$$\tilde{V}^*(s) = \frac{\tilde{V}(s)}{V(1)} = \frac{1}{s} \frac{\sin(\pi s \phi)}{\sin(\pi \phi)},$$

The other parameter is $k = v$ for the pure Harper equation, and $k = v = \pi\phi$ for the general case. If only the first harmonic of the FC is used, the equation shows that a Harper equation in the reciprocal space is also a Harper equation with a renormalized set of parameters [27]. In the case of Fibonacci, Eq. (7) proves that the Schrödinger equation in reciprocal space has a different form, since each site interacts with infinite many others. The interaction between Fourier components decreases as $1/s$. It is a long range interaction that can be thought as a modulating field. Observe that in the transformation between the Hofstadter and the Fibonacci butterfly, the limitation to S harmonics in the potential, is equivalent in Fourier space to a cut-off of range S of the interaction.

When ϕ is a rational number, say P/Q , the potential with any number of harmonics has periodicity Q , as has been discussed in the previous section. The corresponding wave functions are given by,

$$\psi_n = e^{ikn} \sum_{m=0}^Q d_m e^{im(2\pi n \frac{P}{Q} + \pi \frac{P}{Q})}, \quad (9)$$

where the $-\pi/Q < k \leq \pi/Q$. The only difference between the Harper and the Fibonacci case, is in the values of the Q coefficients d_m . Since the set d_m can be obtained in the Harper case using quantum groups [8]. It is worthwhile mentioning that previous efforts in building a perturbation theory to tackle the properties of quasiperiodic Hamiltonians have failed due to the small divisor problem, that shows up in perturbation expansions. The present approach does not have such problem, since a quasiperiodic solution can be built from a perturbation of a solution that is already quasiperiodic.

Also, we can show that an effective potential can be written in the reciprocal space for periodic approximants of the Fibonacci chain. According to Eq. (9), the solution must have periodicity Q , with l an integer. Such result can also be obtained from $V(n)$ when ϕ is the rational P/Q . In this case, the factor $\cos(2\pi s \phi n)$ in Eq.(6) is repeated for the harmonics s' that has the form $s + lQ$ where l is a positive integer. By grouping all the harmonics module Q , the potential is written as,

$$V(n) = \bar{V} + 2\delta V \sum_{s=1}^Q \tilde{V}_{P/Q}(s) \cos\left(2\pi s \frac{P}{Q} n\right), \quad (10)$$

where $\tilde{V}_{P/Q}(s)$ is an effective potential,

$$\tilde{V}_{P/Q}(s) = \frac{Q \sin(\pi s P/Q)}{P} \left(\sum_{l=0}^{\infty} \frac{(-1)^{lP}}{s + lQ} \right). \quad (11)$$

The equation in reciprocal space is reduced as follows,

$$(E' - 2\delta V' \cos(2\pi\phi m + k)) d_m = \quad (12)$$

$$\sum_{s=1}^Q \tilde{V}_{P/Q}(s) (d_{m-s} + d_{m+s}), \quad (13)$$

which shows that for a periodic approximant, the range of the interaction in reciprocal space is Q .

Concerning the spectrum, the band edges are obtained from Eq. (7) when $k = 0$ and $k = \pi/Q$. Instead of following this path, we will look at how the structure of the potential in the *lattice reciprocal space* determines the spectrum for $\delta V \ll 1$. A much more physical insight is obtained in this way. This approach is different from the one realized in others works [4][27], since usually the potential is projected in the base $e^{im(2\pi\phi n + v)}$. Here we will project into the reciprocal "vectors" of the lattice (G). The main idea is that for a one dimensional crystal, it is known that each reciprocal "vector" G with component of the potential ($\tilde{V}(G)$), opens a gap of size,

$$\Delta_G \approx 2 \left\| \tilde{V}(G) \right\|,$$

at reciprocal vectors $q = G/2, G, 3G/2, \dots$. It is possible to follow the opening of the gaps by the effect of $\tilde{V}(G)$. The reciprocal components are,

$$\tilde{V}(G) = \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} V(n) e^{-iGn},$$

where G can be chosen among the wave vectors $q = 2\pi t/N$, with $t = 0, \dots, N-1$, in a lattice with N sites, where periodic boundary conditions are used. The first Brioullin zone is the interval $-\pi \leq q < \pi$, although to simplify the algebra we take for the moment, q between 0 and 2π . Consider first the Fourier component of the Harper potential ($\tilde{V}_H(G)$) for a given parameter ϕ ,

$$\tilde{V}_H(G) \equiv \tilde{V}_H(t) = \frac{2\lambda}{\sqrt{N}} \sum_{n=0}^{N-1} \left(\frac{e^{i2\pi\phi n} + e^{-i2\pi\phi n}}{2} \right) e^{-i2\pi t n/N}. \quad (14)$$

If $(\phi - t/N)$ is an integer, the problem is almost solved, because $\tilde{V}(t) = \lambda\delta(\phi - t/N)$. This happens whenever $\phi = P/Q$ and,

$$\frac{P}{Q} = \frac{m}{N},$$

so when N is chosen as a multiple of Q , as for example $N = lQ$ with $l = 0, 1, 2, \dots$, then the harmonic $m = P$ is the only one that has a contribution, and the problem is solved as a simple cosine potential.

This solution is very simple compared with other complex approaches, since what most of the people do is fix N and then a sweep of the ϕ is made. In other words, the Hofstadter butterfly is built for a fixed N . However, the present approach shows that if we fix a rational ϕ and move N for each ϕ until $N = lQ$, the potential is much more tractable. In the limit of big l , $N \gg Q$ and there are so many vectors k that the condition is very easily achieved; in other words, there is a continuum of q so $e^{i2\pi\phi n}$ has a "possible periodicity" of the system. The Fibonacci case is a little bit more difficult, because the

potential is a sum of cosines, but from Eq. (11) is clear that the components in Fourier space, that we denote by $\tilde{V}_F(G)$, are given by $\tilde{V}_F(G) = \tilde{V}_{P/Q}(t)$.

The problems arise when ϕ is an irrational or N is not a multiple Q , since the previous trick is not valid. Eventually, the sum in Eq. (14) can be made when $\phi - t/N$ is not an integer to give,

$$\tilde{V}_H(G) \equiv \tilde{V}_H(t) = \frac{\lambda}{\sqrt{N}} \left(\frac{1 - e^{i2\pi\phi N}}{1 - e^{i2\pi(\phi - t/N)}} + \frac{1 - e^{-i2\pi\phi N}}{1 - e^{-i2\pi(\phi + t/N)}} \right).$$

The corresponding norm is,

$$\left\| \tilde{V}_H(t) \right\|^2 = \frac{\lambda^2}{N} \frac{A}{B}, \quad (15)$$

where,

$$A = (1 - \cos 2\pi\phi N)^2 + [\cos 2\pi\phi - \cos 2\pi\phi(N-1)]^2 - 2 \cos(2\pi\phi t/N) [1 - \cos 2\pi\phi/N] [\cos 2\pi\phi - \cos 2\pi\phi(N-1)],$$

and,

$$B = [1 - \cos(2\pi(\phi - t/N))] [1 - \cos 2\pi(\phi + t/N)].$$

Notice how $\tilde{V}_H(t) \rightarrow \lambda\delta(\phi - t/N)$ as $\phi \rightarrow t/N$. The gaps widths depend upon these components, and the maximum of $\tilde{V}_H(G)$ occurs when $G \approx \pm\phi$. Now let us propose the solution,

$$\psi_n = \sum_{q=-\infty}^{\infty} c_q e^{iqn},$$

for Eq. (1). The resulting equation in reciprocal space is,

$$(E - 2 \cos q) c_q = \sum_G \tilde{V}_H(G) c_{q-G}$$

Since the main contribution to $\tilde{V}_H(G)$ comes from $G \approx \pm\phi$, a mixing of wave vectors $G \approx \phi$ and $G \approx -\phi$ occurs at $q = \pm\phi/2$. Using perturbation theory, this means that for $\lambda \ll 1$, the main gaps are open around,

$$E \approx \pm\lambda \cos(2\pi(\phi/2)) = \pm\lambda \cos(\pi\phi).$$

Figure 4 compares this prediction with the Hofstadter butterfly, showing an excellent agreement. Other band gaps are obtained at $q = \pm r\phi/2$ with $r \in \mathbb{N}$. The general p th-order perturbation term is of the form,

$$V_q = V(q_0 - q_1)V(q_1 - q_2)\dots V(q_{N-1} - q_N)$$

where q_t labels the vector $2\pi t/N$. For small λ , gaps will be open at,

$$E \approx \lambda^r \cos(\pi\phi r),$$

These cosine branches are also plotted in Figure 4, compared with the Hofstadter butterfly, showing that the

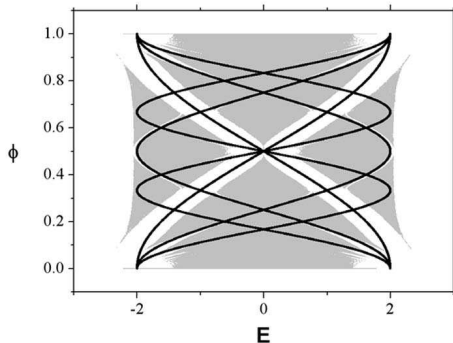


FIG. 4: General structure of all the obtained spectra. The Hofstadter butterfly is compared with the cosine branches given by $E = \lambda \cos(\phi m)$ for $m = 1, 2$ and 3 for $\lambda = 0.2967$.

basic structure of the spectrum is determined by these branches. When λ is near one, around each gap there are many wave vectors that mix together, so the present approximation breaks out.

A detailed observation of the Fibonacci spectrum, shows that the main effect upon the spectrum is an asymmetry of the cosine branches, and the different gap sizes around these branches, due to the mixing of different modulating harmonics. This is very clear in Figures 3 b) and 3 c). The same analysis performed to the Harper equation is valid for these cases, specially for low S . The equivalent rough approximation for the FC is,

$$E \approx \pm \left(\delta V \sum_{s=1}^{\infty} \frac{\sin(\pi s \phi)}{\pi s} \right) \cos(\pi \phi),$$

What is behind the process of gap opening, is a self-similar folding of the bands in the reciprocal space, as revealed by following the sequence of approximants for ϕ in all the obtained spectra. For $\phi = 0$, the spectrum is continuous and goes from $-2\delta V$ to $2\delta V$. The next most simple spectrum corresponds to $\phi = 1/2$, which gives two bands. If the lattice has period 2, then the first Brillouin zone of the zero approximant ($\phi = 0$), is folded around half the original Brillouin zone limit, $k = \pi/2$. As usual, an energy gap of size $\Delta_{\mathbf{G}} \approx 2 |\tilde{V}(\mathbf{G})|$ is open in the zone boundary due to a mixing of waves with reciprocal vectors $\mathbf{G}/2$ and $-\mathbf{G}/2$. In this case, $\mathbf{G} = \pi/2$ and $\tilde{V}(\mathbf{G}) = 2\lambda$. For the Harper equation we get,

$$\Delta_{\mathbf{G}} = 4\lambda,$$

In the Harper equation, there is symmetry around $E = 0$, thus the bands limits are $E = \pm 2\lambda$. For the pure FC first approximant ($\phi = 1/2$), $\tilde{V}(\mathbf{G})$ produces a different gap, and the central gap limits are,

$$\Delta_{\mathbf{G}} = \pm \delta V.$$

This gap is clearly in the horizontal lines at $\phi = 1/2$ in Fig. 3 d). The process of folding in reciprocal space can be repeated in a similar way for other rationals like $\phi = 2/3$ and $\phi = 1/3$. In this case, the periodicity is 3, and the folding around the first Brillouin zone limit occurs at $k = \pi/3$. Three bands are produced in this case, and the gaps are centered at $\pm \lambda \cos(\pi 2/3)$ and $\pm \lambda \cos(\pi/3)$. The process is repeated for other approximants..

IV. LOCALIZATION PROPERTIES USING CORRELATORS

As an example of the utility of having an expansion of the Fibonacci potential, we show how to obtain certain localization properties from the potential correlators. This can serve to understand how the addition of harmonics leads to different localization properties. For $\lambda = 1$ the eigenstates of the Harper equation are critical, while they are localized for $\lambda > 1$ and $\lambda < 1$. This comes from Eq.(7). Therein, if the square of the components is finite,

$$\sum_{m=-\infty}^{\infty} |d_m|^2 < \infty$$

and the corresponding wave-function is non-localized. In the pure Harper equation, when $\lambda \rightarrow \infty$, the solutions in real space are localized wave-functions, since it corresponds to the dual of a solution $\lambda \rightarrow 0$ in reciprocal space, which are known to be extended.

For the Fibonacci chain, it is known that for all values of λ the eigenstates are critical [26]. An interesting question is how many harmonics are needed to produce this transition. Although this requires a complete study, let us show how this problem can be tackled. To measure localization in one dimension, the Lyapunov exponents are used (L^{-1}). They are the inverse of the localization length (L) for an exponential localized state. It has been shown that for one dimensional Hamiltonians, the localization properties depend upon the pair correlation of the potential [28] when $V(n) \ll 1$. Notice that this is the most interesting limit for quasiperiodic potentials, since for $V(n) \gg 1$ there are already many methods to treat the spectrum and the eigenfunctions [29]. For a one dimensional Hamiltonian [28],

$$L^{-1} = \frac{\epsilon_0 \varphi(\mu)}{8 \sin^2(\mu)}; \quad \varphi(\mu) = 1 + 2 \sum_{k=1}^{\infty} \xi(k) \cos(2\mu k)$$

Here, the function $\varphi(\mu)$ is given by the Fourier series with the coefficients $\xi(k)$ which is the correlator of the site potential $V(n)$. The correlator $\xi(k)$ is defined as,

$$\langle V(n)V(n+k) \rangle = \epsilon_0^2 \xi(k), \quad \langle V(n)V(n) \rangle = \epsilon_0^2$$

For the Harper equation, $\xi(k) = \cos(2\pi \phi k)$, which gives $\varphi(\mu) = 0$ and thus all states are non exponentially

localized. In the case of a FC, by using Eq. (6), the correlator can be written as,

$$\xi(k) = 4\delta V \sum_{s=1}^S \frac{\sin^2(\pi s\phi)}{(s\phi)^2} \cos(2\pi s\phi k)$$

which also gives $\varphi(\mu) = 0$ for all S . Thus, we have proved that all states are non exponentially localized for any number of harmonics. The states are extended or critical. An interesting question that remains to be answered, is how the transition from extended to critical states is achieved as the number of harmonics is increased from Harper to Fibonacci. To solve this question, an expression for the scaling exponents in terms of the correlators is needed [30].

V. CONCLUSIONS

In the present article, we have shown that the Fibonacci potential can be written as a sum of Harper

potentials. As a consequence, one can follow the evolution of the spectral types as a function of the number of harmonics. In particular, a butterfly similar to the Hofstadter case is found for the Fibonacci. The corresponding spectrum is a fractal object. The Fourier components of the potential provides a simple explanation for the main features of the spectra. We also show how the development of the potential leads to interesting questions, as for example, at which harmonic the spectral type is changed. This is equivalent to ask for which harmonic the eigenfunctions of the Fibonacci case become critical. The present approach also leads to the possibility of building analytical solutions for a FC. We hope that other researchers will answer some of these interesting questions in the nearby future.

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